ON THE PROBLEM OF INTERACTIONS IN QUANTUM THEORY

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Abstract

The structure of representations describing systems of free particles in the theory with the invariance group SO(1,4) is investigated. The property of the particles to be free means as usual that the representation describing a many-particle system is the tensor product of the corresponding single-particle representations (i.e. no interaction is introduced). It is shown that the mass operator contains only continuous spectrum in the interval $(-\infty, \infty)$ and such representations are unitarily equivalent to ones describing interactions (gravitational, electromagnetic etc.). This means that there are no bound states in the theory and the Hilbert space of the many-particle system contains a subspace of states with the following property: the action of free representation operators on these states is manifested in the form of different interactions. Possible consequences of the results are discussed.

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1 General remarks on quantum theories

The existing quantum theories are usually based on the following procedure: the Lagrangian of the system under consideration is written as $L = L_m + L_g + L_{int}$ where L_m is the Lagrangian

of "matter", L_g is the Lagrangian of gauge fields and L_{int} is the interaction Lagrangian. The symmetry conditions do not define L_{int} uniquely since at least the interaction constant is arbitrary. Nevertheless such an approach has turned out to be highly successful in QED, electroweak theory and QCD. At the same time the difficulties in constructing quantum gravity have not been overcome.

The popular idea expressed by many physicists is that such notions as matter and interactions are not fundamental — they are only manifestations of some properties of space-time (see e.g. Ref. [1]). On the other hand the problem arises what is the meaning of space-time on the quantum level.

Indeed, there is no operator corresponding to time (see e.g. the discussion in Ref. [2]) and the latter is considered only as a classical quantity. Moreover, it has become clear already in 30th that in relativistic quantum theory there is no operator having all the properties of the coordinate operator (see e.g. Ref. [3]). As a consequence, the quantity x in the Lagrangian density $L(x)$ is not the coordinate in Minkowski space but some parameter which becomes the coordinate only in the classical limit.

One can consider the Lagrangian only as an auxiliary tool for constructing all the generators of the symmetry group in the framework of the canonical formalism (see e.g. Ref. [4]). Nevertheless the practical realization of such an approach encounters serious mathematical problems. One of the reason is that the interacting field operators can be treated only as operator valued distributions [5] and therefore the product of two local field operators at coinciding points is not well defined. The problem of the correct definition of such products is known as the problem of constructing composite operators (see e.g. Ref. [6]). So

far this problem has been solved only in the framework of perturbation theory for special models. When perturbation theory does not apply the usual prescriptions are to separate the arguments of the operators in question and to define the composite operator as a limit of nonlocal operators when the separation goes to zero (see e.g. Ref. [7] and references therein). However the Lagrangian contains products of local operators at the same point. If one separates the arguments of these operators, the Lagrangian immediately becomes nonlocal and it is not clear how to use the Noether procedure in this case.

As a consequence, the generators constructed in a standard way are usually not well-defined (see e.g. Ref. [8]) and the theory contains anomalies. It is also worth noting that there exists the well-known paradox: on the one hand the renormalizable perturbation theory, which is formally based on interaction picture, is in beautiful agreement with many experimental data, while on the other hand, according to the Haag theorem [9, 5], the interaction picture does not exist.

One of the advantages of superstring theories is that they involve products of operators at different points and there exist theories without anomalies [10]. At the same time the theories involve such notions as Lagrangian, interaction and perturbation theory.

In chiral theories (see e.g. Ref. [11] and references therein) the interaction Lagrangian is usually not introduced but the fields have the range in some nonlinear manifold. There exist serious problems in quantizing such theories.

The above remarks show that at present physicists have no clear answer to the questions whether in quantum theory the notions of space-time, Lagrangian and interaction are primary

or they are manifestations of some deeper reasons. According to the Heisenberg program (which was very popular in 60th) such notions are rudiments which will not survive in the future theory (see e.g. the discussion in Ref. [12]). According to this program, the only fundamental operator is the S-operator defined on the Hilbert space of elementary particles while bound states are manifested only as poles in the matrix elements of this operator. It is also interesting to pose the following problem. Is it possible that the ultimate physical theory is fully defined by the choice of the symmetry group and there is no need to introduce the Lagrangian and interactions?

In the present paper we consider a model which in our opinion can shed light on this problem. We choose the de Sitter group SO(1,4) (more precisely its covering group $\overline{SO(1,4)}$) as the symmetry group. We require as usual that the elementary particles are described by unitary irreducible representations (UIRs) of this group and different realizations of such representations are described in Sect. 2. Then we assume that the representation describing the many-particle system is the tensor product of the corresponding single-particle representations. According to the usual philosophy this means that the particles are free and no interaction is introduced. In Sect. 3 we explicitly calculate the free many-particle mass operator and show that the spectrum of this operator contains the whole interval $(-\infty, \infty)$. As shown in Sect. 4, such an operator is unitarily equivalent to the mass operator containing interactions (gravitational, strong, electromagnetic etc.). Finally, Sect. 5 is discussion.

It is worth noting that the unusual properties of SO(1,4)-invariant theories considered in this paper are specific only for these theories while SO(2,3)-invariant theories have many com-

mon features with Poincare-invariant ones (the mass of the elementary particle coincides with the minimal value of its energy, the mass of the two-particle system has the minimal value $m_1 + m_2$ etc. [13]).

2 Realizations of single-particle representations of the SO(1,4) group

The de Sitter group SO(1,4) is the symmetry group of the four-dimensional manifold which can be described as follows. If $(x_0, x_1, x_2, x_3, x_4)$ are the coordinates in the five-dimensional space, the manifold is the set of points satisfying the condition

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -R_0^2 \tag{1}$$

where $R_0 > 0$ is some parameter. Elements of a map of the point $(0, 0, 0, 0, R_0)$ (or $(0, 0, 0, 0, -R_0)$) can be parametrized by the coordinates (x_0, x_1, x_2, x_3). If R_0 is very large then such a map proceeds to Minkowski space and the action of the de Sitter group on this map — to the action of the Poincare group. The quantity R_0^2 is often written as $R_0^2 = 3/\Lambda$ where Λ is the cosmological constant. The existing astronomical data show that Λ is very small and the usual estimates based on popular cosmological models give $R_0 > 10^{26} cm$. On the other hand, in models based on the de Sitter cosmology the quantity R_0 is related to the Hubble constant H as $R_0 = 1/H$ and in this case R_0 is of order $10^{27} cm$ [14].

The representation generators of the SO(1,4) group L^{ab} $(a, b = 0, 1, 2, 3, 4,\ L^{ab} = -L^{ba})$ should satisfy the commutation

relations

$$[L^{ab}, L^{cd}] = -\imath(\eta^{ac}L^{bd} + \eta^{bd}L^{as} - \eta^{ad}L^{bc} - \eta^{bc}L^{ad}) \qquad (2)$$

where η^{ab} is the diagonal metric tensor such that $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = -\eta^{44} = 1$.

In conventional quantum theory elementary particles are described by UIRs of the symmetry group or its Lie algebra. If one assumes that the role of the symmetry group is played by the Poincare group, then the representations are described by ten generators — six generators of the Lorentz group and the four-momentum operator. In the units $c = \hbar = 1$ the former are dimensionless while the latter has the dimension $(length)^{-1}$. If however the symmetry group is the de Sitter group SO(1,4), then all the generators in the units $c = \hbar = 1$ are dimensionless. There exists wide literature devoted to the UIRs of this group (see e.g. Refs. [15, 16, 17, 18]). Below we describe three different realizations of the UIRs. The reader can explicitly verify that the generators indeed satisfy Eq. (2).

If s is the spin of the particle under consideration, then we use $||...||$ to denote the norm in the space of the UIR of the group SU(2) with the spin s. Let $v = (v_0 = (1 + \mathbf{v}^2)^{1/2}, \mathbf{v})$ be the element of the Lorentz hyperboloid of four-velocities and dv be the Lorentz invariant volume element on this hyperboloid. Then one can realize the UIR under consideration in the space of functions $\{f_1(v), f_2(v)\}$ on two Lorentz hyperboloids with the range in the space of the UIR of the group SU(2) with the spin s and such that

$$\int [||f_1(v)||^2 + ||f_2(v)||^2] dv < \infty \qquad (3)$$

The explicit calculation shows that the action of the generators

on $f_1(v)$ has the form

$$\mathbf{M} = l(\mathbf{v}) + \mathbf{s}, \quad \mathbf{N} == -\imath v_0 \frac{\partial}{\partial \mathbf{v}} + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},$$
$$\mathbf{B} = \mu\mathbf{v} + \imath[\frac{\partial}{\partial \mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial \mathbf{v}}) + \frac{3}{2}\mathbf{v}] + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},$$
$$L_{04} = \mu v_0 + \imath v_0(\mathbf{v}\frac{\partial}{\partial \mathbf{v}} + \frac{3}{2}) \tag{4}$$

where $\mathbf{M} = \{L^{23}, L^{31}, L^{12}\}$, $\mathbf{N} = \{L^{01}, L^{02}, L^{03}\}$, $\mathbf{B} = -\{L^{14}, L^{24}, L^{34}\}$, $\mathbf{s}$ is the spin operator, and $\mathbf{l}(\mathbf{v}) = -\imath\mathbf{v}\times\partial/\partial\mathbf{v}$. The action of L^{ab} on $f_2(v)$ is obtained from Eq. (4) by the substitution $\mu \to -\mu$.

Such a realization is used to obtain the possible closest analogy between the representations of the SO(1,4) and the Poincare group. It is easy to see that the operators $\mathbf{M}$ and $\mathbf{N}$ in Eq. (4) have the same form as for the standard realization of the single-particle representations of the Poincare group (see e.g. Ref. [19, 20]) while the contraction of the representation Eq. (4) into the standard realization of the Poincare group is accomplished as follows. Denote $m = \mu/R$, $\mathbf{P} = \mathbf{B}/R$ and $E = L_{04}/R$ where $R > 0$ is some quantity of order R_0 (but not necessarily equal to R_0). Consider the action of the generators on functions which do not depend on R in the usual system of units. Then, as follows from Eqs. (1) and (4), in the limit $R \to \infty$ we obtain the standard representation of the Poincare group for a particle with the mass m, since $\mathbf{P} = m\mathbf{v}$, $E = mv_0$ (in this case one has the representation of the Poincare group with the negative energy on the second hyperboloid).

Since the representation generators of the SO(1,4) group are dimensionless (in the units $c = \hbar = 1$), any quantal descrip-

tion in the SO(1,4)-invariant theory involves only dimensional quantities. In particular, as seen from Eq. (4), the quantal description of particles in such a theory does not involve any information about the quantity R_0 (this property is clear from the fact that the elements of the SO(1,4) group describe only homogeneous transformations of the manifold defined by Eq. (1)). The latter comes into play only when we wish to interpret the results in terms of quantities used in the Poincare-invariant theory. Therefore if we assume that de Sitter invariance is fundamental and Poincare invariance is only approximate, it is reasonable to think that the de Sitter masses μ are fundamental while the quantities m, R_0 and R are not (see also the discussion in Ref. [21]). In particular, R (or even R_0) may be time-dependent (see below) and in this case the usual masses will be time-dependent too.

It is also possible to realize the UIR in the space of functions $\varphi(u)$ on the three-dimensional unit sphere S^3 in the four-dimensional space with the range in the space of the UIR of the group SU(2) with the spin s and such that

$$\int ||\varphi(u)||^2 du < \infty \tag{5}$$

where du is the SO(4) invariant volume element on S^3. Elements of S^3 can be represented as $u = (\mathbf{u}, u_4)$ where $u_4 = \pm(1-\mathbf{u}^2)^{1/2}$ for the upper and lower hemispheres respectively. Then the explicit calculation shows that the generators for this realization have the form

$$\mathbf{M} = l(\mathbf{u}) + \mathbf{s}, \quad \mathbf{B} = \imath u_4 \frac{\partial}{\partial \mathbf{u}} - \mathbf{s},$$
$$\mathbf{N} = \imath[\frac{\partial}{\partial \mathbf{u}} - \mathbf{u}(\mathbf{u}\frac{\partial}{\partial \mathbf{u}})] - (\mu + \frac{3\imath}{2})\mathbf{u} + \mathbf{u}\times\mathbf{s} - u_4\mathbf{s},$$

$$L_{04} = (\mu + \frac{3\imath}{2})u_4 + \imath u_4 \mathbf{u}\frac{\partial}{\partial \mathbf{u}} \tag{6}$$

Since Eqs. (3) and (4) on the one hand and Eqs. (5) and (6) on the other are the different realization of one and the same representation, then there exists a unitary operator transforming functions $f(v)$ into $\varphi(u)$ and operators (4) into operators (6). For example in the spinless case

$$\varphi(u) = exp(-\frac{\imath}{2}\,\mu\, lnv_0)v_0^{3/2}f(v) \tag{7}$$

where $\mathbf{u} = -\mathbf{v}/v_0$. In view of this relation, the sphere S^3 is usually interpreted in the literature as the velocity space (see e.g. Refs. [16, 17]), but, as argued in Ref. [18], there are serious arguments to interpret S^3 as the coordinate space. Below we give additional arguments in favor of this point of view.

As follows from Eq. (1), if x_0 is fixed then the set of the points satisfying this relation is the three-dimensional sphere $S^3(R_1)$ with the radius $R_1 = (R_0^2 + x_0^2)^{1/2}$. This sphere is invariant under the action of the SO(4) subgroup of the SO(1,4) group. The operators $\mathbf{B}$ and $\mathbf{M}$ are the representation generators of the SO(4) subgroup. We can choose $\mathbf{x} = R_1\mathbf{u}$ as the coordinates on $S^3(R_1)$. In these coordinates the operators $\mathbf{B}$ and $\mathbf{M}$ given by Eq. (6) are the generators of the representation of the group of motions of $S^3(R_1)$ induced from the representation of the SO(3) subgroup with the generators $\mathbf{s}$.

Consider a vicinity of the south pole of $S^3(R_1)$ such that $|\mathbf{x}| \ll R_1$. Then the generators in Eq. (6) have the form

$$\mathbf{B} = R_1\mathbf{p}, \quad \mathbf{M} = \mathbf{l}(\mathbf{x}) + \mathbf{s}, \quad \mathbf{N} = -R_1\mathbf{p}, \quad L_{04} = mR_1 \tag{8}$$

where $\mathbf{p} = -\imath\partial/\partial\mathbf{x}$ and $m = \mu/R_1$. Therefore $\mathbf{B}/R_1$ is the de

Sitter analog of the momentum operator, but L_{04}/R_1 in this realization is not the de Sitter analog of the energy operator.

The reason of such a situation is as follows. In Poincare invariant theories one can consider wave functions defined on the conventional three-dimensional space R^3. The operators $\mathbf{M}$ and $\mathbf{P}$ are the representation generators of the group of motions of R^3. From the remaining generators, E and $\mathbf{N}$, only E commutes with $\mathbf{M}$ and $\mathbf{P}$. For this reason E can be chosen as the operator responsible for the evolution of the system under consideration while stationary states are the eigenstates of E. In the SO(1,4) case one can consider wave functions defined on $S^3(R_1)$ at different values of x_0. However none of the generators $L_{04}, \mathbf{N}$ commutes with all the operators $\mathbf{M}$ and $\mathbf{B}$. At the same time the operator $E_{dS} = (L_{04}^2 + \mathbf{N}^2)^{1/2}$ satisfies this property. Hence E_{dS} can be treated as the operator responsible for the evolution. At the conditions described by Eq. (8), $E_{dS} = R_1(m^2 + \mathbf{p}^2)^{1/2}$ and therefore E_{dS} can be considered as the de Sitter analog of the energy operator.

The inconvenience of working with $\mathbf{B}$ as the de Sitter analog of the momentum operator is that different components of $\mathbf{B}$ commute with each other only when $R_1 \to \infty$. We can define the operators $\mathbf{Q}_+ = (L_{1+}, L_{2+}, L_{3+})$ and $\mathbf{Q}_- = (L_{1-}, L_{2-}, L_{3-})$, where the $\pm$ components of five-vectors are defined as $x_\pm = x_4 \pm x_0$. Then, as follows from Eq. (2), different components of $\mathbf{Q}_+$ commute with each other and the same is valid for $\mathbf{Q}_-$.

It is easy to see that $2\mathbf{u}/(1-u_4)$ is the stereographic projection of the point $(\mathbf{u}, u_4) \in S^3$ onto the three-dimensional space. Now we will use $\mathbf{x}$ to denote the quantity $\mathbf{x} = 2R\mathbf{u}/(1-u_4)$ where R is some quantity of order R_0 and $R = R_1$ is a reasonable choice. In the space of functions $\varphi(\mathbf{x})$ on R^3 with the range

in the space of the UIR of the group SU(2) with the spin s and such that

$$\int ||\varphi(\mathbf{x})||^2 d^3\mathbf{x} < \infty \tag{9}$$

the generators of the UIR of the SO(1,4) group can be realized as follows

$$\begin{aligned}
&\mathbf{M} = \mathbf{l}(\mathbf{x}) + \mathbf{s}, \quad L_{+-} = -2(\mu + \mathbf{x}\mathbf{p}) + 3\imath, \quad \mathbf{Q}_+ = -2R\mathbf{p}, \\
&\mathbf{Q}_- = \frac{1}{2R}[-2\mu\mathbf{x} + \mathbf{x}^2\mathbf{p} - 2\mathbf{x}(\mathbf{x}\mathbf{p}) + 3\imath\mathbf{x} + 2(\mathbf{s}\times\mathbf{x})]
\end{aligned} \tag{10}$$

where we again use $\mathbf{p} = -\imath\partial/\partial\mathbf{x}$, but the quantities $\mathbf{x}$ and $\mathbf{p}$ are not the same as in Eq. (8).

The UIR realized by Eqs. (4), (6) or (10) belongs to the so called principal series. It can be characterized by the condition that $\mu^2 \geq 0$, i.e. μ is real. In contrast with the UIRs of the Poincare group, the sign of μ does not make it possible to distinguish the UIRs describing particles and antiparticles (see e.g. the discussion in Ref. [16]), and the UIRs with μ and $-\mu$ are unitarily equivalent.

The Casimir operator of the second order can be written as

$$\begin{aligned}
I_2 = -\sum_{ab} L_{ab}L^{ab} = \frac{1}{2}(L_{+-})^2 - \\
(\mathbf{Q}_+\mathbf{Q}_- + \mathbf{Q}_-\mathbf{Q}_+) - 2\mathbf{M}^2
\end{aligned} \tag{11}$$

A direct calculation shows that in the case described by Eqs. (4), (6) and (10)

$$I_2 = 2(\mu^2 - \mathbf{s}^2 + \frac{9}{4}) \tag{12}$$

In Poincare invariant theories the spectrum of the mass operator can be defined as the spectrum of the energy operator in

the subspace of states with zero total momentum. As follows from Eq. (10), for the UIRs of the SO(1,4) group corresponding to the principal series, the spectrum of the mass operator can be defined from the condition that the action of L_{+-} on the states with zero momentum is equal to $-2\mu+3\imath$. The presence of $3\imath$ in this expression does not contradict the Hermiticity of L_{+-} since L_{+-} does not commute with $\mathbf{Q}_+$.

3 Free many-particle mass operator in the SO(1,4)-invariant theory

The representation describing a system of N noninteracting particles is constructed as the tensor product of the corresponding single-particle representations, and the representation generators are equal to the sums of single-particle generators, i.e.

$$L_{ab} = \sum_{n=1}^{N} L_{ab}^{(n)} \tag{13}$$

where $L_{ab}^{(n)}$ are the generators for the nth particle. Each generator acts through the variables of its "own" particle, as described in the preceding section, and through the variables of other particles it acts as the identity operator.

The tensor product of single-particle representations can be decomposed into the direct integral of UIRs and there exists the well elaborated general theory [22]. In the given case, among the UIRs there may be not only representations of the principal series but also UIRs of other series.

We first consider the case of two particles 1 and 2. Suppose that the UIRs for them are realized as in Eq. (4). Introduce the conventional masses and momenta $m_j = \mu_j/R$, $\mathbf{p}_j = m_j\mathbf{v}_j$

($j = 1,2$). We can define the variables describing the system as a whole and the internal variables. The usual nonrelativistic variables are:

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{k} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2} \tag{14}$$

Then, in the approximation when the particle velocities are very small, it follows from Eqs. (4) and (13) that the two-particle generators have the form

$$\begin{aligned}
&\mathbf{M} = \mathbf{l}(\mathbf{P}) + \mathbf{S}, \quad \mathbf{N} = -\imath(m_1 + m_2)\frac{\partial}{\partial\mathbf{P}},\\
&L_{04} = R(m_1 + m_2) + \imath(\mathbf{k}\frac{\partial}{\partial\mathbf{k}} + \frac{3}{2}) + \imath(\mathbf{P}\frac{\partial}{\partial\mathbf{P}} + \frac{3}{2}),\\
&\mathbf{B} = R\mathbf{P} + \imath(m_1 + m_2)\frac{\partial}{\partial\mathbf{P}}
\end{aligned} \tag{15}$$

where $\mathbf{S} = \mathbf{l}(\mathbf{k}) + \mathbf{s}_1 + \mathbf{s}_2$. The comparison of the expressions (4) and (15) for $\mathbf{M}$ shows that $\mathbf{S}$ plays the role of the spin operator for the system as a whole (in full analogy with the conventional quantum mechanics).

By analogy with Eq. (12) we can define the mass operator M_{dS} for the system as a whole. Namely, if I_2 is the Casimir operator for the system as a whole defined by this expression, then

$$I_2 = 2(M_{dS}^2 - \mathbf{S}^2 + \frac{9}{4}) \tag{16}$$

In turn, the conventional mass operator M can be defined as M_{dS}/R.

As follows from Eqs. (15) and (16), for slow particles in first order in $1/R$

$$M = m_1 + m_2 + \frac{\imath}{R}(\mathbf{k}\frac{\partial}{\partial\mathbf{k}} + \frac{3}{2}) \tag{17}$$

We shall see below that this expression is correct for any velocities and in any order in $1/R$ if only the representations of the principal series are taken into account.

Equation (17) means that for very slow particles the de Sitter correction to the classical nonrelativistic Hamiltonian is equal to $\Delta H_{nr} = (\mathbf{kr})/R$ where $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ is the vector of the relative distance between the particles (this quantity is conjugated with $\mathbf{k}$). As follows from the classical equations of motion, $\ddot{\mathbf{r}} = \mathbf{r}/R^2$. Therefore the correction corresponds to the well-known fact that in the classical SO(1,4)-invariant theory there exists the antigravity, and the force of (cosmological) repulsion between particles is proportional to the distance between them. It is also interesting to note that the de Sitter antigravity is in some sense even more universal than the usual gravity: the force of repulsion does not depend on the parameters characterizing the particles (even on their masses).

Now we again consider the case of two particles but suppose that the UIRs for them are realized as in Eq. (10). Introduce the standard nonrelativistic variables

$$\mathbf{X} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} = \frac{\mu_1\mathbf{x}_1 + \mu_2\mathbf{x}_2}{\mu_1 + \mu_2}, \quad \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2 \qquad (18)$$

Then a direct calculation of the two-particle generators gives:

$$L_{+-} = -2(\mu_1 + \mu_2 + \mathbf{XP}) + 2\imath(\mathbf{r}\frac{\partial}{\partial\mathbf{r}} + 3),$$
$$\mathbf{Q}_+ = -2R\mathbf{P}, \quad \mathbf{M} = \mathbf{l}(\mathbf{X}) + \mathbf{S},$$
$$\mathbf{Q}_- = -(m_1 + m_2)\mathbf{X} + \frac{1}{2R}[\mathbf{X}^2\mathbf{P} + \frac{m_1m_2}{(m_1+m_2)^2}\mathbf{r}^2\mathbf{P} -$$
$$2\imath(\mathbf{rX}\frac{\partial}{\partial\mathbf{r}} - \imath\frac{m_2 - m_1}{m_1 + m_2}\mathbf{r}^2\frac{\partial}{\partial\mathbf{r}} - 2\mathbf{X}(\mathbf{XP}) -$$

$$\frac{2m_1m_2}{(m_1+m_2)^2}\mathbf{r}(\mathbf{r}\mathbf{P}) + 2\imath\mathbf{X}(\mathbf{r}\frac{\partial}{\partial\mathbf{r}}) + 2\imath\mathbf{r}(\mathbf{X}\frac{\partial}{\partial\mathbf{r}}) +$$
$$\frac{2\imath(m_2-m_1)}{m_1+m_2}\mathbf{r}(\mathbf{r}\frac{\partial}{\partial\mathbf{r}}) + 6\imath\mathbf{X} + \frac{3\imath(m_2-m_1)}{m_1+m_2}\mathbf{r}] +$$
$$\frac{1}{R}[(\mathbf{s}_1+\mathbf{s}_2)\times\mathbf{X} + \frac{m_2\mathbf{s}_1 - m_1\mathbf{s}_2}{m_1+m_2}\times\mathbf{r}] \tag{19}$$

where $\mathbf{S} = \mathbf{l}(\mathbf{r}) + \mathbf{s}_1 + \mathbf{s}_2$ and $\mathbf{P} = -\imath\partial/\partial\mathbf{X}$.

A direct calculation shows that, as a consequence of Eqs. (11), (16) and (19),

$$M_{dS}^2 = [\mu_1 + \mu_2 - \imath(\mathbf{r}\frac{\partial}{\partial\mathbf{r}} + \frac{3}{2})]^2 + \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2}[\mathbf{r}^2\mathbf{P}^2 -$$
$$2(\mathbf{r}\mathbf{P})^2] + \frac{\imath(\mu_2 - m_1)}{\mu_2+\mu_1}[2(\mathbf{r}\mathbf{P})(\mathbf{r}\frac{\partial}{\partial\mathbf{r}}) + 3\mathbf{r}\mathbf{P} - \mathbf{r}^2(\mathbf{P}\frac{\partial}{\partial\mathbf{r}})] +$$
$$\frac{4(\mu_2\mathbf{s}_1 - \mu_1\mathbf{s}_2)(\mathbf{r}\times\mathbf{P})}{\mu_1+\mu_2} \tag{20}$$

$$M^2 = [m_1 + m_2 - \frac{\imath}{R}(\mathbf{r}\frac{\partial}{\partial\mathbf{r}} + \frac{3}{2})]^2 + \frac{m_1m_2}{R^2(m_1+m_2)^2}[\mathbf{r}^2\mathbf{P}^2 -$$
$$2(\mathbf{r}\mathbf{P})^2] + \frac{\imath(m_2 - m_1)}{R^2(m_2+m_1)}[2(\mathbf{r}\mathbf{P})(\mathbf{r}\frac{\partial}{\partial\mathbf{r}}) + 3\mathbf{r}\mathbf{P} - \mathbf{r}^2(\mathbf{P}\frac{\partial}{\partial\mathbf{r}})] +$$
$$\frac{4(m_2\mathbf{s}_1 - m_1\mathbf{s}_2)(\mathbf{r}\times\mathbf{P})}{R^2(m_1+m_2)} \tag{21}$$

The expressions for both M_{dS} and M have been explicitly written down in order to stress that if M_{dS} is expressed in terms of the de Sitter masses then it does not depend on R. Such a dependence arises only when we consider the conventional mass operator in terms of conventional masses (see the discussion in the preceding section).

As follows from Eq. (21), the decomposition of the tensor product of two UIRs belonging to the principal series contains not only UIRs belonging to this series. Indeed, the spectrum of the operator M^2 is not positive definite. This is clear, for example, from the fact that for very large values of $|\mathbf{P}|$ and the values of $\mathbf{r}$ collinear with $\mathbf{P}$, the value of M^2 becomes negative. However if R is very large then such values of $|\mathbf{P}|$ are practically impossible. It is obvious from Eq. (21) that for realistic values of $|\mathbf{P}|$ the contribution to M^2 of the UIRs not belonging to the principal series is a small correction of order $1/R^2$.

If only the contribution of UIRs belonging to the principal series is taken into account, then the problem of determining M^2 can be considered as follows. Since the tensor product of two UIRs can be decomposed into the direct integral of UIRs [22] and any UIR of the principal series is unitarily equivalent to the representation (10) with some operators $\mathbf{s}$ and values of μ, we can conclude that any representation of the SO(1,4) group containing only the UIRs of the principal series is unitarily equivalent to the representation defined by the generators

$$\begin{aligned}&\mathbf{M} = \mathbf{l}(\mathbf{X}) + \mathbf{S}, \quad L_{+-} = -2(M_{dS} + \mathbf{X}\mathbf{P}) + 3\imath,\\ &\mathbf{Q}_+ = -2R\mathbf{P}, \quad \mathbf{Q}_- = \frac{1}{2R}[-2M_{dS}\mathbf{X} + \mathbf{X}^2\mathbf{P} -\\ &2\mathbf{X}(\mathbf{X}\mathbf{P}) + 3\imath\mathbf{X} + 2(\mathbf{S}\times\mathbf{X})]\end{aligned} \tag{22}$$

where the generators $\mathbf{S}$ and M_{dS} act only through the internal variables of the system under consideration. By analogy with the case of UIRs considered in the preceding section, it is clear that M_{dS} can be determined by considering the action of L_{+-} on the states with $\mathbf{P} = 0$: the action of L_{+-} on such states is equal to $2(-M_{dS} + 3\imath/2)$ (recall that the sign of M_{dS} does not play a

role). Therefore, as follows from Eq. (19), the mass operator in the given case is unitarily equivalent to the operator

$$M = m_1 + m_2 - \frac{\imath}{R}(\mathbf{r}\frac{\partial}{\partial \mathbf{r}} + \frac{3}{2}) \qquad (23)$$

In particular, the positive part of the operator (21) is unitarily equivalent to the square of the operator given by Eq. (23). This is in agreement with the "common wisdom" according to which the spectrum of the mass operator is defined by its reduction on the (generalized) subspace of states with $\mathbf{P} = 0$.

The comparison of Eqs. (17) and (23) is an additional argument in treating $S^3(R_1)$ as the coordinate space, at least at very small velocities (note that $\mathbf{r} = \imath\partial/\partial\mathbf{k}$, $\mathbf{k} = -\imath\partial/\partial\mathbf{r}$). We will use momentum and coordinate representations depending on convenience. In the first case the operator (17) acts in the space of functions $f(\mathbf{k})$ such that

$$\int |f(\mathbf{k})|^2 d^3\mathbf{k} < \infty \qquad (24)$$

and in the second case — in the space of functions $\varphi(\mathbf{r})$ such that

$$\int |\varphi(\mathbf{r})|^2 d^3\mathbf{r} < \infty \qquad (25)$$

(here and henceforth we will consider only the spinless case for simplicity). The functions $f(\mathbf{k})$ and $\varphi(\mathbf{r})$ are the Fourier transforms of each other.

In spherical coordinates $\mathbf{r}\partial/\partial\mathbf{r} = r\partial/\partial r$ where $r = |\mathbf{r}|$. Therefore in these coordinates the operator (23) does not act through the angular variables. We can consider the action of this operator in the space of functions $\varphi(r)$ such that

$$\int |\varphi(r)|^2 r^2 dr < \infty \qquad (26)$$

The eigenvalue problem

$$(m_1+m_2)\varphi_\lambda(r)-\frac{\imath}{R}[r\frac{d\varphi_\lambda(r)}{dr}+\frac{3}{2}\varphi_\lambda(r)]=\lambda\varphi_\lambda(r) \tag{27}$$

has the solution

$$\varphi_\lambda(r)=\frac{1}{r}(\frac{R}{2\pi r})^{1/2}exp[\imath R(\lambda-m_1-m_2)ln\,r] \tag{28}$$

where we assume that r is given in some dimensional units. Then

$$\int_0^\infty \varphi_\lambda(r)^*\varphi_{\lambda'}(r)r^2dr=\delta(\lambda-\lambda'),$$
$$\int_{-\infty}^\infty \varphi_\lambda(r)^*\varphi_\lambda(r')d\lambda=\frac{1}{r^2}\delta(r-r') \tag{29}$$

where * means the complex conjugation. Therefore the operator (23) contains only the continuous spectrum occupying the interval $(-\infty,\infty)$. The same is obviously valid for the operator (17). As a consequence, we have

Statement 1: *The operators* (17) *and* (23) *are unitarily equivalent to the operator*

$$M_0=\frac{\imath}{R}(\mathbf{k}\frac{\partial}{\partial\mathbf{k}}+\frac{3}{2})=-\frac{\imath}{R}(\mathbf{r}\frac{\partial}{\partial\mathbf{r}}+\frac{3}{2}) \tag{30}$$

When $R\to\infty$ we must have the compatibility of the above results with the standard results of the Poincare invariant theory, according to which the mass operator is given by $M^P=(m_1^2+k^2)^{1/2}+(m_2^2+k^2)^{1/2}$ where $k=|\mathbf{k}|$. We use $g(k^2)$ to denote the function

$$\begin{aligned} g(k^2) \;&=\; \frac{1}{2}\int_0^{k^2}\{[(m_1^2+k'^2)^{1/2}+m_1]^{-1}+ \\ &\quad [(m_2^2+k'^2)^{1/2}+m_2]^{-1}\}dk'^2 \end{aligned} \tag{31}$$

Then it is obvious that

$$(m_1^2+k^2)^{1/2}+(m_2^2+k^2)^{1/2}+\frac{\imath}{R}(\mathbf{k}\frac{\partial}{\partial\mathbf{k}}+\frac{3}{2})=$$
$$exp[\imath Rg(k^2)][m_1+m_2+\frac{\imath}{R}(\mathbf{k}\frac{\partial}{\partial\mathbf{k}}+\frac{3}{2})]$$
$$exp[-\imath Rg(k^2)] \tag{32}$$

We can formulate this result as

Statement 2: *The operator*

$$\tilde{M}=(m_1^2+k^2)^{1/2}+(m_2^2+k^2)^{1/2}+\frac{\imath}{R}(\mathbf{k}\frac{\partial}{\partial\mathbf{k}}+\frac{3}{2}) \tag{33}$$

is unitarily equivalent to the operator given by Eq. (17).

If one considers the action of the operator $\tilde{M}$ on functions $f(\mathbf{k})$ satisfying the conditions

$$|\partial f(\mathbf{k})/\partial\mathbf{k}|\ll R|f(\mathbf{k})| \tag{34}$$

then the action of $\tilde{M}$ is practically indistinguishable from the action of M^P.

In the approximation when only the representations containing the UIRs of the principal series are considered, the above results can be easily generalized to the case of any number of particles. Indeed, suppose that the system under consideration consists of two subsystems, α and β. Let M_α and M_β be the corresponding mass operators and $\mathbf{k}_{\alpha\beta}$ be the relative momentum. Then, by analogy with Eq. (17),

$$M_{\alpha\beta}=M_\alpha+M_\beta+\frac{\imath}{R}(\mathbf{k}_{\alpha\beta}\frac{\partial}{\partial\mathbf{k}_{\alpha\beta}}+\frac{3}{2}) \tag{35}$$

By analogy with the above consideration it is easy to show that this operator is unitarily equivalent to

$$\tilde{M}_{\alpha\beta} = (M_\alpha^2 + k_{\alpha\beta}^2)^{1/2} + (M_\beta^2 + k_{\alpha\beta}^2)^{1/2} + \frac{\imath}{R}(\mathbf{k}_{\alpha\beta}\frac{\partial}{\partial \mathbf{k}_{\alpha\beta}} + \frac{3}{2}) \quad (36)$$

where $k_{\alpha\beta} = |\mathbf{k}_{\alpha\beta}|$. Therefore when R is large, the Hilbert space of the internal wave functions contains a subspace of functions with the following property: the action of $\tilde{M}_{\alpha\beta}$ on these functions is practically indistinguishable from the action of the standard mass operator $(M_\alpha^2+k_{\alpha\beta}^2)^{1/2}+(M_\beta^2+k_{\alpha\beta}^2)^{1/2}$. It is also clear that the cosmological repulsion has a place for any pair of particles in the system under consideration.

Consider, for example, a system of three particles with the masses m_1, m_2 and m_3, and introduce the standard Jacobi variables

$$\mathbf{k}_{12} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1+m_2}, \quad \mathbf{K}_{12} = \frac{m_3(\mathbf{p}_1+\mathbf{p}_2)-(m_1+m_2)\mathbf{p}_3}{m_1+m_2+m_3} \quad (37)$$

Then it follows from Eqs. (17) and (35) that

$$M_{123} = m_1+m_2+m_3+\frac{\imath}{R}(\mathbf{k}_{12}\frac{\partial}{\partial \mathbf{k}_{12}}+\frac{3}{2})+\frac{\imath}{R}(\mathbf{K}_{12}\frac{\partial}{\partial \mathbf{K}_{12}}+\frac{3}{2}) \quad (38)$$

This operator is unitarily equivalent to

$$\tilde{M}_{123} = (M_{12P}^2 + K_{12}^2)^{1/2} + (m_3^2 + K_{12}^2)^{1/2} + \frac{\imath}{R}(\mathbf{k}_{12}\frac{\partial}{\partial \mathbf{k}_{12}} + \frac{3}{2}) + \frac{\imath}{R}(\mathbf{K}_{12}\frac{\partial}{\partial \mathbf{K}_{12}} + \frac{3}{2}) \quad (39)$$

where M_{12}^P is the mass operator of the system (12) in the Poincare invariant theory. There exists a subspace of functions $f(\mathbf{k}_{12}, \mathbf{K}_{12})$

with the following property: the action of $\tilde{M}_{123}$ on these functions is practically indistinguishable from the action of the standard mass operator $(M_{12P}^2 + K_{12}^2)^{1/2} + (m_3^2 + K_{12}^2)^{1/2}$. The functions $f(\mathbf{k}_{12}, \mathbf{K}_{12})$ should satisfy the property

$$|\frac{\partial f(\mathbf{k}_{12}, \mathbf{K}_{12})}{\partial \mathbf{k}_{12}}|, |\frac{\partial f(\mathbf{k}_{12}, \mathbf{K}_{12})}{\partial \mathbf{K}_{12}}| \ll R|f(\mathbf{k}_{12}, \mathbf{K}_{12})| \tag{40}$$

4 Unitary equivalence of free and interacting representations in the SO(1,4)-invariant theory

If the particles in the system under consideration interact with each other then the representation generators describing this system are interaction dependent but they must satisfy the commutation relations (2). By analogy with the procedure proposed by Bakamdjian and Thomas in Poincare-invariant theories [23], we can introduce the interaction by replacing the free mass operator M_{dS} in Eq. (22) by an interacting mass operator $\hat{M}_{dS}$. Then the relations (22) will be obviously satisfied if $\hat{M}_{dS}$ acts only through the internal variables and commutes with $\mathbf{S}$.

In the general case the spin and momentum operators in Eq. (22) can be interaction dependent too but, by analogy with Poincare-invariant theories (see e.g. Ref. [20]) it is natural to assume that they have the same spectrum as the corresponding free operators. Therefore one can eliminate the interaction dependence of the spin and momentum operators by using a proper unitary transformation. In Poincare-invariant theories the corresponding unitary operators are known as Sokolov's packing operators (see e.g. Refs. [24, 25, 26, 20, 27, 28]).

In the present paper we will consider the representation generators in the Bakamdjian-Thomas (BT) form, but the above discussion gives grounds to think that any representation describing the interacting system in the SO(1,4)-invariant theory and taking into account only the contribution of the UIRs of the principal series is unitarily equivalent to the representation in the BT form.

Consider first in detail the case of two free particles in the nonrelativistic approximation. We can write $\tilde{M} - m_1 - m_2$ in the form

$$M_{nr} = \frac{k^2}{2m_{12}} + \frac{\imath}{R}(\mathbf{k}\frac{\partial}{\partial \mathbf{k}} + \frac{3}{2}) \tag{41}$$

where $m_{12} = m_1 m_2/(m_1 + m_2)$ is the reduced mass of particles 1 and 2. In coordinate representation this operator has the form

$$M_{nr} = -\frac{\Delta}{2m_{12}} - \frac{\imath}{R}(\mathbf{r}\frac{\partial}{\partial \mathbf{r}} + \frac{3}{2}) \tag{42}$$

where $\Delta = (\partial/\partial \mathbf{r})^2$.

It is obvious that

$$M_{nr} = exp(\frac{\imath R k^2}{4m_{12}})M_0 exp(-\frac{\imath R k^2}{4m_{12}}) \tag{43}$$

where M_0 is given by Eq. (30). Therefore we have

Statement 3: *The operators M_0 and M_{nr} are unitarily equivalent.* In particular, M_{nr} contains only the continuos spectrum occupying the interval $(-\infty, \infty)$.

The Hilbert space of functions satisfying the conditions (24) or (25) can be decomposed into the subspaces H_{lm} such that the elements of H_{lm} have the form

$$f(\mathbf{k}) = Y_{lm}(\mathbf{k}/k)f(k), \quad \varphi(\mathbf{r}) = Y_{lm}(\mathbf{r}/r)\varphi(r) \tag{44}$$

Y_{lm} is the spherical function, $\varphi(r)$ satisfies the condition (26) and $f(k)$ satisfies the analogous condition in momentum representation.

In this representation the eigenvalue problem for the operator M_{nr} in H_{lm} does not depend on l and m:

$$\frac{k^2}{2m_{12}}f_\lambda(k) + \frac{\imath}{R}[k\frac{df_\lambda(k)}{k} + \frac{3}{2}f_\lambda(k)] = \lambda f_\lambda(k) \tag{45}$$

The solution of this equation is

$$f_\lambda(k) = \frac{1}{k}(\frac{R}{2\pi k})^{1/2}exp[\imath R(\frac{k^2}{4m_{12}} - \lambda ln\, k)] \tag{46}$$

In coordinate representation the eigenvalue problem for the operator M_{nr} in H_{lm} reads

$$\begin{aligned} &-\frac{1}{2m_{12}r^2}\frac{d}{dr}[r^2\frac{d\varphi_{\lambda l}(r)}{dr}] + \frac{l(l+1)}{2m_{12}r^2}\varphi_{\lambda l}(r) - \\ &-\frac{\imath}{R}[r\frac{d\varphi_{\lambda l}(r)}{dr} + \frac{3}{2}\varphi_{\lambda l}(r)] = \lambda\varphi_{\lambda l}(r) \end{aligned} \tag{47}$$

The relation between the functions $f_\lambda(k)$ and $\varphi_{\lambda l}(r)$ is given by the radial Fourier transform:

$$\varphi_{\lambda l}(r) = \frac{R^{1/2}}{\pi}(-\imath)^l \int_0^\infty j_l(kr)k^{1/2}exp[\imath R(\frac{k^2}{4m_{12}} - \lambda ln\, k)]dk \tag{48}$$

where

$$j_l(kr) = (\frac{\pi}{2kr})^{1/2}J_{l+1/2}(kr) \tag{49}$$

is the spherical Bessel function.

The integral in Eq. (48) can be calculated analytically. We use γ^2 to denote $-\imath R/4m_{12}$. Then [29]

$$\varphi_{\lambda l}(r) = (\frac{R}{2\pi r})^{1/2}\frac{(-\imath)^l}{2\Gamma(l+3/2)}\gamma^{\imath\lambda R-1}(\frac{r}{2\gamma})^{l+1/2}$$

$$\Gamma(\frac{l}{2}+\frac{3}{4}-\frac{\imath\lambda R}{2})_1F_1(\frac{l}{2}+\frac{3}{4}-\frac{\imath\lambda R}{2};l+\frac{3}{2};\frac{r^2}{4\gamma^2}) \qquad (50)$$

where Γ is the gamma function and $_1F_1$ is the hypergeometric function (in Ref. [29] a general case is considered and one has to require $Re(\gamma^2) > 0$ but in our case it is also possible to use the result of Ref. [29] if $Re(\gamma^2) = 0$).

As follows from Eq. (50), when $r \to 0$, the function $\varphi_{\lambda l}(r)$ is proportional to r^l. This fact was clear from the analogy with the conventional quantum mechanics. Indeed, by analogy with the standard investigation of the radial Schroedinger equations, one can expect that at $r \to 0$ the first two terms in Eq. (47) are dominant. It is well-known that the only regular solution at such conditions is proportional to r^l.

Let us now consider the asymptotic behavior of $\varphi_{\lambda l}(r)$ when $r \to \infty$. It is convenient to use not Eq. (50) but the original integral (48). Introducing the new integration variable $t = kr$ and using Eq. (49) one arrives at the following asymptotic result:

$$\begin{aligned}\varphi_{\lambda l}(r) &= \frac{(-\imath)^l}{r}(\frac{R}{2\pi r})^{1/2}exp(\imath\lambda R) \\ &\quad \int_0^\infty J_{l+1/2}(t)exp(-\imath R\lambda ln\, t)dt \qquad (51)\end{aligned}$$

Since [29]

$$\begin{aligned}&\int_0^\infty J_{l+1/2}(t)exp(-\imath R\lambda ln\, t)dt = \\ &2^{-\imath R\lambda}\Gamma(\frac{l}{2}+\frac{3}{4}-\frac{\imath\lambda R}{2})/\Gamma(\frac{l}{2}+\frac{3}{4}+\frac{\imath\lambda R}{2}) \qquad (52)\end{aligned}$$

the comparison of Eqs. (47), (51) and (52) with Eqs. (27) and (28) shows that at $r \to \infty$ one can neglect the first two terms in Eq. (47).

We can normalize the functions $\varphi_{\lambda l}(r)$ as

$$\int_0^\infty \varphi_{\lambda l}(r)^* \varphi_{\lambda l}(r) r^2 dr = \delta(\lambda - \lambda') \tag{53}$$

and any function $\varphi(\mathbf{r})$ from the internal Hilbert space can be written as

$$\varphi(\mathbf{r}) = \sum_{lm} \int_{-\infty}^{\infty} c_{lm}(\lambda) Y_{lm}(\mathbf{r}/r) \varphi_{\lambda l}(r) d\lambda \tag{54}$$

Now we proceed to the case of interacting particles and consider the operator

$$\hat{M}_{nr} = -\frac{\Delta}{2m_{12}} + V(r) - \frac{\imath}{R}(\mathbf{r}\frac{\partial}{\partial \mathbf{r}} + \frac{3}{2}) \tag{55}$$

The eigenvalue problem for this operator in H_{lm} has the form

$$-\frac{1}{2m_{12}r^2}\frac{d}{dr}[r^2\frac{d\psi_{\lambda l}(r)}{dr}] + \frac{l(l+1)}{2m_{12}r^2}\psi_{\lambda l}(r) + V(r)\psi_{\lambda l}(r) -$$
$$\frac{\imath}{R}[r\frac{d\psi_{\lambda l}(r)}{dr} + \frac{3}{2}\psi_{\lambda l}(r)] = \lambda\psi_{\lambda l}(r) \tag{56}$$

Suppose that $V(r)r^2 \to 0$ when $r \to 0$. Then the third term in Eq. (55) is negligible in comparison with the first two ones when $r \to 0$ (see e.g. Ref. [30]). Therefore the asymptotic behavior of the function $\psi_{\lambda l}(r)$ at $r \to 0$ is the same as that of the function $\varphi_{\lambda l}(r)$, i.e. $\psi_{\lambda l}(r)$ is proportional to r^l. On the other hand, if $V(r) \to 0$ when $r \to \infty$ then the third term in Eq. (55) is negligible in comparison with the fourth one when $r \to \infty$. Therefore the asymptotic behavior of the function $\psi_{\lambda l}(r)$ at $r \to \infty$ also coincides with that of the function $\varphi_{\lambda l}(r)$, i.e. the function $\psi_{\lambda l}(r)$ at $r \to \infty$ is proportional to the expression given by Eq. (51).

Since the functions $\psi_{\lambda l}(r)$ and $\varphi_{\lambda l}(r)$ have the same asymptotic behavior at $r \to 0$ and $r \to \infty$, we conclude that the operators M_{nr} and $\hat{M}_{nr}$ have the same spectrum. Since the normalization of the eigenfunctions belonging to the continuous spectrum is fully determined by the asymptotic behavior of these functions at $r \to \infty$ [30], we can normalize the functions $\psi_{\lambda l}(r)$ in the same way as the functions $\varphi_{\lambda l}(r)$:

$$\int_0^\infty \psi_{\lambda l}(r)^* \psi_{\lambda l}(r) r^2 dr = \delta(\lambda - \lambda') \tag{57}$$

Then we can define the operator U as follows. If the function $\varphi(\mathbf{r})$ is given by Eq. (54) then

$$U\varphi(\mathbf{r}) = \sum_{lm} \int_{-\infty}^{\infty} c_{lm}(\lambda) Y_{lm}(\mathbf{r}/r) \psi_{\lambda l}(r) d\lambda \tag{58}$$

The operator U commutes with $\mathbf{S}$ by construction. As follows from Eqs. (53) and (57), this operator is unitary, and $\hat{M}_{nr} = UM_{nr}U^{-1}$. Therefore we have

Statement 4: The operators $\hat{M}_{nr}$ and M_{nr} are unitarily equivalent.

In the nonrelativistic approximation we have to consider functions $f(\mathbf{k})$ for which the important values of $\mathbf{k}$ are much smaller than the masses of the particles in question. Suppose also that R is very large and the functions satisfy the condition (34). In coordinate representation the actions of M_{nr} and $\hat{M}_{nr}$ on such functions are practically indistinguishable from the actions of the operators

$$M_{nr}^G = -\frac{\Delta}{2m_{12}} \quad \text{and} \quad \hat{M}_{nr}^G = -\frac{\Delta}{2m_{12}} + V(r), \tag{59}$$

respectively ("G" means "Galilei"). However the operators M_{nr}^G and $\hat{M}_{nr}^G$ are not necessarily unitarily equivalent. For example, if

$V(r) = -const/r$ and $const > 0$ then the operator M_{nr}^G has only the continuous spectrum in the interval $[0, \infty)$ while $\hat{M}_{nr}^G$ has also the discrete spectrum at some negative values of λ. Since the operators M_{nr}^G and $\hat{M}_{nr}^G$ in this case have different spectra, they cannot be unitarily equivalent.

The result formulated in *Statement 4* could be expected from physical considerations. Indeed, if $V(r)$ is not too singular when $r \to 0$, the phenomenon known as the "fall onto the center" (see e.g. Ref. [30]) does not occur and the spectra of M_{nr} and $\hat{M}_{nr}$ are defined by asymptotic of the eigenfunctions of these operators at $r \to \infty$. Even if R is very large, there exist such values of r that the cosmological repulsion becomes dominant in comparison with the kinetic and potential energies. Since this repulsion is present in both M_{nr} and $\hat{M}_{nr}$, these operators have the same spectrum and therefore are unitarily equivalent.

Consider now a system of N particles with arbitrary velocities and suppose that the interactions between the particles can be described only in terms of the degrees of freedom characterizing these particles. The gravitational and electromagnetic interactions are not very singular in the sense that they fall off at infinity and do not lead to the fall onto the center [30]. In the case of strong interactions the problem exists how to describe the interaction of colored objects at large distances. Such an interaction is often modelled by attractive potentials which at infinity are proportional to r. In this case the force of attraction does not depend on r and therefore can be neglected in comparison with the cosmological repulsion. Therefore it is natural to say that at infinity all realistic interactions are negligible in comparison with the cosmological repulsion (or by definition,

the necessary condition for any interaction to be realistic is to be small in comparison with the cosmological repulsion when $r \to \infty$). For these reasons, in the case of realistic interactions, asymptotic of the eigenfunctions of the interacting mass operator is again defined by the cosmological repulsion and is the same as asymptotic of the eigenfunctions of the free mass operator discussed in the preceding section. Therefore we can formulate the following

Statement 5: In the SO(1,4)-invariant theory the interacting mass operator of the system of N particles described by the UIRs of the principal series is unitarily equivalent to the free mass operator.

If the unitary operator realizing the equivalence of two mass operators commutes with **S** and the corresponding representations can be realized in the BT form then they are unitarily equivalent.

In QED, electroweak theory and QCD the Hilbert space for the system under consideration is the Fock space describing a system of infinite number of particles. As noted in Sect. 1, there exist serious mathematical problems in constructing the representation operators of the Poincare group in these theories. The problem becomes much more complicated if the symmetry group is the group of motions of a curved space-time (see e.g. Ref. [31]). The above considerations give grounds to think that in all SO(1,4)-invariant theories of realistic interactions the spectrum of the representation generators is fully defined by the cosmological repulsion which is dominant at large distances. Therefore we can formulate the following

Conjecture: In any SO(1,4)-invariant theory of realistic interactions the interacting and free representation generators are

unitarily equivalent.

5 Discussion

A standard problem of the perturbation theory for linear operators (see e.g. Ref. [32]) is as follows. Let A and $\hat{A}$ be selfadjoint operators in the Hilbert space. Suppose that they have the same (absolutely) continuous spectrum (this is treated as the property of $\hat{A}$ to be in some sense a small perturbation of A). Suppose also for simplicity that A does not contain other points of the spectrum. Consider the wave operators

$$W_{\pm}(A,\hat{A}) = s - lim_{t\to\pm\infty}\ exp(\imath\hat{A}t)exp(-\imath At) \tag{60}$$

where $s{-}lim$ means the strong limit. If $\hat{A}$ has the same spectrum as A then there exist conditions when the operators $W_{\pm}$ are unitary and

$$\hat{A} = W_{\pm}(A,\hat{A})AW_{\pm}(A,\hat{A})^{-1} \tag{61}$$

i.e. A and $\hat{A}$ are unitarily equivalent. If $\hat{A}$ also contains the discrete spectrum, the operators A and $\hat{A}$ cannot be unitarily equivalent but there exist conditions when the operators $W_{\pm}$ are isometric and the S-operator $S = W_+^* W_-$ is unitary.

As shown in the preceding section, in the SO(1,4)-invariant theory the interacting mass operator $\hat{M}$ of a many-particle system has the same spectrum as the free mass operator M and these operators are unitarily equivalent. The absence of bound states is a consequence of the fact that at large distances the cosmological repulsion is dominant. The choice of the unitary operator realizing the equivalence of $\hat{M}$ and M is obviously not unique. By analogy with the standard results of the perturbation theory for linear operators one could expect that a possible choice is $W_{\pm}(M,\hat{M})$.

If R is very large and one considers only the subspace of H^P of functions satisfying the conditions analogous to (34), (40) etc. then the actions of $\hat{M}$ and M on these functions are practically indistinguishable from the actions of the corresponding operators in the Poincare-invariant theory (obtained from M and $\hat{M}$ by neglecting the cosmological repulsion). Therefore in the SO(1,4)-invariant theory there exist quasi-bound states: their lifetime is very large and goes to infinity when $R \to \infty$. It is clear that there exist conditions when the quasi-bound states are practically indistinguishable from bound ones. The finite lifetime is related to the fact that theoretically there exists a nonzero probability for quasi-bound particles to pass through the barrier separating the usual and cosmological distances. However in practice this probability can be negligible.

It is important to note that the subspace H^P, where the results of the Poincare-invariant theory are valid, is only a small part of the full Hilbert space H. In particular, if $f \in H^P$ then $exp(\imath\hat{M}t)$ and $exp(-\imath Mt)$ cannot belong to H^P if t is of order R/c. Therefore it is natural to think that the standard scattering problem in H^P is meaningful only if $ct \ll R$.

The above remarks may also be considered as an indication that difficulties of the present quantum theory (e.g. divergencies) are related to the fact that loop contributions to the S-operator involve not only a set of states belonging to H^P but a much wider set of states for which it is not possible to neglect the effects of de Sitter invariance. In other words, these effects can play a role of regularizers for usual divergent expressions in the standard approach. For the case of the SO(2,3) symmetry such a possibility has been considered in Ref. [33].

If *Conjecture* formulated in the preceding section is valid then it is natural to think that the very notion of interaction is not fundamental. Indeed, in this case *there is no need to introduce interaction terms into the operators: we can always work with the free operators and physics is defined by the subset of states important in the processes under consideration.*

The subsets corresponding to different interactions are connected with each other by unitary transformations which necessarily depend on R. Indeed, if one reduces the free and interacting operators onto H^P and neglects the cosmological repulsion, then, as noted above, the operators obtained in such a way are not unitarily equivalent in the general case.

In particular, one might think that for the situation corresponding to the pair of the operators M_{nr} and $\hat{M}_{nr}$ (see the preceding section) the really fundamental problem is not the choice of the potential $V(r)$ which should be added to M_{nr} but the choice of the unitary operator U realizing the unitary equivalence of M_{nr} and $\hat{M}_{nr}$. In the framework of such an approach one might think that the really fundamental quantities are those defining the operator U. In this case the gravitational constant, electric charges etc. are functions of more fundamental quantities and R, in agreement with the famous Dirac hypothesis [34] about the dependence of physical constants on cosmological parameters.

Moreover, in view of the above discussion it is natural to think that all the existing interactions are fully defined by the present state vector of the Universe. This can be treated as the quantum analog of Mach's principle according to which the local physical laws are defined by the distribution of masses in the Universe (the discussion of Mach's principle and its relation

to general relativity and Dirac's cosmology can be found in wide literature — see e.g. Ref. [35] and references therein).

In summary, SO(1,4)-invariant theories have rather unusual properties, in particular the mass operator has only continuous spectrum in the interval $(-\infty, \infty)$, bound states do not exist and the representations describing free and interacting systems are unitarily equivalent. At the same time SO(1,4) invariance does not contradict the existing experimental data and therefore there exists the possibility that the SO(1,4) group is the symmetry group of the nature. For these reasons the investigation of SO(1,4)-invariant theories is of indubitable interest.

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